3.6 (continued)

Variation of Parameters

$$y'' + p(t)y' + q(t)y = g(t)$$

Solution: $y = c_1 y_1 + c_2 y_2 + Y$ $\qquad$ y_1, y_2 fundamental solutions

$$y = u_1 y_1 + u_2 y_2$$

$$\boxed{\begin{aligned} u_1' y_1 + u_2' y_2 &= 0 \\ u_1' y_1' + u_2' y_2' &= g(t) \end{aligned}}$$

$$\cancel{\begin{bmatrix} u_1' \\ u_2' \end{bmatrix}} \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ g \end{bmatrix}$$

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ g \end{bmatrix}$$

$$= \frac{1}{W} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix} \begin{bmatrix} 0 \\ g \end{bmatrix}$$

$$\boxed{u_1' = -\frac{y_2 g}{W}} \qquad \boxed{u_2' = \frac{y_1 g}{W}}$$

example $y'' - y = e^t + e^{-t}$

$r^2 - 1 = 0 \quad r = \pm 1 \quad y_1 = e^t \quad y_2 = e^{-t}$

$y = c_1 e^t + c_2 e^{-t} + Y$

if we used undetermined coeff.

$Y = Ae^t + Be^{-t}$ but it copies y_1 and y_2

fix: $Y = Ate^t + Bte^{-t}$

but if we use Variation, the method takes care of the copying issue

we just simply solve the system

$$u_1' y_1 + u_2' y_2 = 0$$
$$u_1' y_1' + u_2' y_2' = g$$

$$u_1' e^{t} + u_2' e^{-t} = 0$$
$$u_1' e^{t} - u_2' e^{-t} = e^{t} + e^{-t}$$

add them

$$2u_1' e^{t} = e^{t} + e^{-t}$$
$$u_1' = \frac{1}{2} + \frac{1}{2} e^{-2t}$$

$$\boxed{u_1 = \frac{1}{2} t - \frac{1}{4} e^{-2t} + c_1}$$

$$u_1' e^{t} + u_2' e^{-t} = 0$$

$$u_2' e^{-t} = -u_1' e^{t} = -\left(\frac{1}{2} + \frac{1}{2} e^{-2t}\right) e^{t}$$
$$= -\frac{1}{2} e^{t} - \frac{1}{2} e^{-t}$$
$$u_2' = -\frac{1}{2} e^{2t} - \frac{1}{2}$$

$$\boxed{u_2 = -\frac{1}{4} e^{2t} - \frac{1}{2} t + c_2}$$

general solution : $y = u_1 y_1 + u_2 y_2$

$$y = \left(\tfrac{1}{2}t - \tfrac{1}{4}e^{-2t} + c_1\right)e^{t} + \left(-\tfrac{1}{4}e^{2t} - \tfrac{1}{2}t + c_2\right)e^{-t}$$

$$= c_1 e^{t} + c_2 e^{-t} \underbrace{- \tfrac{1}{4}e^{t}} \; \underline{- \tfrac{1}{4}e^{-t}} + \tfrac{1}{2}te^{t} - \tfrac{1}{2}te^{-t}$$

absorb

because

$c_1 e^t - \frac{1}{4}e^t$

$= (c_1 - \frac{1}{4})e^t = c_1 e^t$

the t's we would need to supply in undet. coeff

$$= \boxed{\underbrace{c_1 e^{t} + c_2 e^{-t}}_{\text{complementary}} + \underbrace{\tfrac{1}{2}te^{t} - \tfrac{1}{2}te^{-t}}_{\text{particular}}}$$

example $y'' - 4y' + 4y = e^{2t}$

$$\underbrace{y'' - 4y' + 4y}$$

$r^2 - 4r + 4 = 0 \qquad (r-2)^2 = 0 \qquad r = 2,\ 2$

fundamental solutions: $y_1 = e^{2t} \qquad y_2 = te^{2t}$

this we need to supply

$$u_1' y_1 + u_2' y_2 = 0$$
$$u_1' y_1' + u_2' y_2' = g$$

this time, let's do $u_1' = -\dfrac{y_2 g}{W}$, $u_2' = \dfrac{y_1 g}{W}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2t} & te^{2t} \\ 2e^{2t} & 2te^{2t} + e^{2t} \end{vmatrix}$$

$$= 2te^{4t} + e^{4t} - 2te^{4t} = e^{4t}$$

$$u_1' = -\frac{y_2 g}{W} = -\frac{te^{2t} \cdot e^{2t}}{e^{4t}} = -t \qquad u_1 = -\frac{1}{2}t^2 + c_1$$

$$u_2' = \frac{y_1 g}{W} = \frac{e^{2t} \cdot e^{2t}}{e^{4t}} = 1 \qquad u_2 = t + c_2$$

$$y = u_1 y_1 + u_2 y_2$$

$$= \left(-\frac{1}{2}t^2 + c_1\right)e^{2t} + (t + c_2)te^{2t}$$

$$= c_1 e^{2t} + c_2 te^{2t} - \frac{1}{2}t^2 e^{2t} + t^2 e^{2t}$$

$$= \boxed{c_1 e^{2t} + c_2 te^{2t} + \frac{1}{2}t^2 e^{2t}}$$

variation can handle non constant coefficients, provided we have y_1 and y_2 (somehow).

<u>example</u> $\underbrace{t^2y'' - 3ty' + 4y}_{\text{Euler's equation or Euler-Cauchy eq.}} = t^2\ln(t) \quad (t > 0)$

solutions are $y = t^r$ instead of e^{rt}

here, $y_1 = t^2$

$y_2 = t^2\ln(t)$

for Euler's eq. we throw $\ln(t)$ at solution if things are copying each other

solve $u_1'y_1 + u_2'y_2 = 0$

$u_1'y_1' + u_2'y_2' = g$

$$u_1' t^2 + u_2' t^2 \ln(t) = 0$$

$$2u_1' t + u_2' (t + 2t\ln(t)) = t^2 \ln(t)$$

$$u_1' = -u_2' \ln(t)$$

$$-2u_2' t\ln(t) + t u_2' + 2u_2' t \ln(t) = t^2 \ln(t)$$

$$u_2' = t\ln(t)$$

$$\boxed{u_2 = \frac{1}{2}t^2 \ln(t) - \frac{1}{4} + c_2}$$

$$u_1' = -t[\ln(t)]^2$$

$$\boxed{u_1 = \frac{1}{4}t^2 - \frac{1}{2}t^2 \ln(t) + \frac{1}{2}t^2 (\ln(t))^2 + c_1}$$

$$y = u_1 y_1 + u_2 y_2$$

Variation can be used for higher-order eqs. and even 1st-order

$$t y' + 2y = \frac{\cos(t)}{t}$$

$$y' + \frac{2}{t} y = \frac{\cos(t)}{t^2}$$ normally done w/ integrating factor

but variation works too

first, solve $y' + \frac{2}{t} y = 0$ (separable)

$\vdots$

$y = c \cdot \frac{1}{t^2}$ fundamental / complementary

replace c w/ u : $y = u(t) \cdot \frac{1}{t^2}$ plug into diff. eq.

$$= ut^{-2}$$

$$y' = -2ut^{-3} + u't^{-2}$$

$$-2ut^{-2} + u't^{-1} + 2ut^{-2} = \frac{\cos(t)}{t}$$

$u' = \cos(t)$ $\qquad$ $u = \sin(t) + C$

solution: $y = u \cdot \frac{1}{t^2} = (\sin(t) + C)\frac{1}{t^2}$

$$y = \frac{\sin(t)}{t^2} + \frac{C}{t^2}$$

same as if done w/ integrating factor